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Best L^1-Approximation by Polynomials

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1. INTRODUCTION

For any integer $n \geqslant 0$ and any real $f \in L^1(I)$ with $I = [-1, 1]$, let $E_n(f)$ denote the error of the best L_1-approximation to f with polynomials of degree not exceeding n (let P_n denote the set of all such polynomials). We are interested in upper and lower estimates of $E_{n-1}(f)$ and particularly in its asymptotic behaviour as n tends to infinity. In the literature we found, besides the special function $f(x) = x^n$, only two classes of functions where these questions have been answered to some extent: Estimates and asymptotic behaviour are given in [2, p. 318–319] and in [4, p. 42] for functions like $f(x) = (a - x)^s$ with real s and real $a > 1$. On the other hand, the order of $E_{n-1}(f)$ is determined by the inequality (see [9, p. 84])

$$m_n = \inf |f^{(n)}(x)| \leqslant 2^{n-1} n!\, E_{n-1}(f) \leqslant \sup |f^{(n)}(x)| = M_n,$$

up to a factor M_n/m_n. Under additional assumptions on $f^{(n)}$ there exist asymptotic results for the corresponding L_∞-error [6, p. 79].

According to a theorem of Markoff (see Theorem 2 below), for a wide class of functions (including every f with monotonic $(n-1)$th derivative), the error $E_{n-1}(f)$ is given by

$$E_{n-1}(f) = \left| \int_{-1}^{1} f(t)\, sg_n(t)\, dt \right|, \tag{1.1}$$

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where $sg_n(t)$ is a well-known unimodular function (see (2.2) below). The weakness of (1.1) stems from the fact that the smallness of the right-hand side of (1.1) depends on cancellation. If $f^{(n)}$ is continuous, then partial integration or Peano's therem [3, p. 69] yields

$$\int_{-1}^{1} f(t)\, sg_n(t)\, dt = (-1)^n \cdot \int_{-1}^{1} V_n(t) f^{(n)}(t)\, dt, \tag{1.2}$$

with

$$V_n(t) = \frac{1}{(n-1)!} \int_{-1}^{t} (t-u)^{n-1} \cdot sg_n(u)\, du. \tag{1.3}$$

The function $V_n(t)$ turns out to be positive on $(-1, 1)$ (Theorem 1), and therefore (1.1) and (1.2) imply for every f with nonnegative $f^{(n)}$ that

$$E_{n-1}(f) = \int_{-1}^{1} V_n(t) f^{(n)}(t)\, dt, \tag{1.4}$$

with a nonnegative integrand. If $f^{(n)}$ changes sign we still have the general inequality (see Theorem 2)

$$\left| \int_{-1}^{1} V_n(t) f^{(n)}(t)\, dt \right| \leqslant E_{n-1}(f) \leqslant \int_{-1}^{1} V_n(t)\, |f^{(n)}(t)|\, dt.$$

In order to answer the above-posed questions with the aid of (1.4), we approximate $V_n(t)$ by simpler functions. Thus, we show (see Theorem 5) that $V_n(t)$ is of the same magnitude as

$$\tilde{V}_n(t) = \frac{1}{n!} \left(\frac{1-t^2}{2} \right)^n \cdot \{1 + \sqrt{(n+1)(1-t^2)}\}, \tag{1.5}$$

and that $V_n(t)$ deviates from (see (2.5) for the definition of γ_n)

$$V_n^*(t) = \frac{2\sqrt{2}}{\pi} \cdot \frac{(n+1)}{n!} \gamma_n \cdot \left(\frac{1-t^2}{2} \right)^{n+1/2} \tag{1.6}$$

by less than $(n+1)\, 8^{1-n}/n!$. As application we will handle functions like e^{ax}, e^{ax^2}, $\cos \omega x$, x^{n+m}, and $(x-t)_+^{n-1}$. Typical results are

$$2^{n-1} \cdot n!\, E_{n-1}(e^x) = 1 + 1/4n + O(n^{-2}) \qquad \text{for} \quad n \to \infty$$

and

$$E_{2n-2}(e^{x^2}) = E_{2n-1}(e^{x^2}) = \frac{2^{1-2n}}{n!} \sqrt{e} \cdot \{1 + O(n^{-1})\} \qquad \text{for} \quad n \to \infty.$$

Many of our results depend on a representation of $V_n(t)$ by means of an integral (Theorem 4). We found it remarkable that this integral yields an explicit representation of certain trigonometric sums. Thus, we obtain, for example,

$$\sum_{l=-[n/2]}^{[n/2]} (-1)^l \left(\cos\frac{\pi l}{n+1}\right)^n = \frac{2^{1-n}(n+1)}{\pi}\cdot\int_{-1}^{1}\frac{(1-u^2)^n\,du}{1+(-1)^n u^{2(n+1)}} \sim 2^{1-n}\cdot\sqrt{n/\pi}. \tag{1.7}$$

2. Notations

In the sequel n resp. k denotes a natural number resp. integer. The Chebysheff-polynomial of the second kind

$$U_n(x) = \sin(n+1)\,\Theta/\sin\Theta; \qquad x = \cos\Theta$$

has the n zeros $x_1, x_2, \ldots, x_n$, where

$$x_k = x_{n,k} = -\cos\frac{\pi k}{n+1}, \qquad \text{for} \quad k \in \mathbb{Z}. \tag{2.1}$$

Let

$$sg_n(t) = (-1)^n \operatorname{sgn} U_n(t), \tag{2.2}$$

where

$$\begin{aligned} \operatorname{sgn} y &= y/|y| && \text{if} \quad y \neq 0 \\ &= 0 && \text{if} \quad y = 0; \end{aligned}$$

then

$$sg_n(-t) = (-1)^n\, sg_n(t), \qquad \text{and} \qquad sg_n(-1) = 1, \tag{2.3}$$

and (see [9, p. 72; 2, p. 94])

$$\begin{aligned} \int_{-1}^{1} t^k sg_n(t)\,dt &= 0 && \text{for} \quad 0 \leqslant k < n \\ &= (-1)^n\cdot 2^{1-n} && \text{for} \quad k = n. \end{aligned} \tag{2.4}$$

For $\alpha \geqslant 0$ let

$$\gamma_\alpha = \int_{-1}^{1}(1-u^2)^\alpha\,du = B(1/2, \alpha+1) = \frac{\sqrt{\pi}\,\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)}. \tag{2.5}$$

We deduce

$$\gamma_0 = 2; \qquad \gamma_{1/2} = \pi/2; \qquad (\alpha + 1)\,\gamma_\alpha \gamma_{\alpha+1/2} = \pi. \tag{2.6}$$

Equation (2.5) and Stirling's formula [1, p. 257; #6.1.39] imply $\alpha^{1/2}\gamma_\alpha \to \pi^{1/2}$ as $\alpha \to \infty$, and therefore

$$4/3 \leqslant \alpha^{1/2}\gamma_\alpha \leqslant \pi^{1/2} \qquad \text{for} \quad \alpha \geqslant 1, \tag{2.7}$$

since $\alpha^{1/2}\gamma_\alpha$ increases for $\alpha \geqslant 1$.

We write $x_+ = \max\{x, 0\}$ and define the function

$$F(\sigma) = \sum_{k=-\infty}^{\infty} (-1)^k \exp(-\pi^2 k^2/\sigma) \qquad \text{for} \quad \sigma > 0. \tag{2.8}$$

This representation of $F(\sigma)$ is advantageous for $\sigma \leqslant 1$ and yields, for example, $F(\sigma) \geqslant 1 - 2\exp(-\pi^2/\sigma) \geqslant \frac{1}{2}$ for $\sigma \leqslant 1$. Using properties of the theta-function [10, p. 476] we obtain

$$F(\sigma) = \sqrt{\sigma/\pi}\, e^{-\sigma/4} \cdot \sum_{k=-\infty}^{\infty} e^{-k(k+1)\sigma} > \sqrt{\sigma/\pi}\, e^{-\sigma/4}. \tag{2.9}$$

For any $f \in L_1(I)$ let $p_f = p_{f,n-1}$ denote the polynomial of P_{n-1} that interpolates f at the zeros of $U_n(x)$. M_n stands for the set of all $f \in L^1(I)$ for which $(f - p_f)\, sg_n$ does not change sign on I, and F_n (resp. F_n^+) denotes the set of all $f \in L^1(I)$ that are representable in the form

$$f(x) = p(x) + \int_{-1}^{x} \frac{(x-t)^{n-1}}{(n-1)!}\, dg(t),$$

where $g(t)$ is of bounded variation (resp. increasing) on I and $p(\cdot) \in P_{n-1}$. Note that $f \in F_n^+$ means that $f^{(n-1)}$ increases ($f^{(n-2)}$ being absolutely continuous), and we have

$$F_n^+ \subset M_n. \tag{2.10}$$

This is trivial if $n = 1$. Let therefore $n \geqslant 2$. We may suppose $p(x) \equiv 0$ and $g(-1) = 0$. We may further suppose that $g(t)$ is strictly increasing (the general case then follows by approximating $g(t)$ by $g(t) + \varepsilon(t-1)$). Then $f - p_f$ has exactly n roots, each of which is simple. Otherwise

$$\int_{-1}^{x} g(t)\, dt - (a_1 x + b) = f^{(n-2)}(x) - p_f^{(n-2)}(x)$$

would have at least three roots $y_1 < y_2 < y_3$. This would imply $1/(y_2 - y_1) \int_{y_1}^{y_2} g(t)\, dt = 1/(y_3 - y_2) \int_{y_2}^{y_3} g(t)\, dt$, and thus $g(t) \equiv$ constant on (y_1, y_3) since

$g(t)$ increases. We remark that $(f-p_f)\operatorname{sgn} U_n \geqslant 0$ for $f \in F_n^+$. This follows from the special example $f(x)=x^n$ by means of a homotopy argument.

3. Some Elementary Results

From (1.3) we deduce for $t \in I$:

$$V_n^{(k)}(t) = \frac{1}{(n-k-1)!} \cdot \int_{-1}^{t} (t-u)^{n-k-1}\, sg_n(u)\, du \quad \text{for} \quad 0 \leqslant k < n,$$

$$= sg_n(t) \qquad \text{for} \quad k=n \text{ and } t \neq x_1, x_2, ..., x_n, \tag{3.1}$$

and this yields immediately statements about the zeros of $V_n(t)$ and its derivatives.

Theorem 1 (Properties of the Peano-kernel $V_n(t)$).

(a) *$V_n(t)$ is an even function.*

(b) *$\forall\, 0 \leqslant k < n$: $V_n^{(k)}(t)$ has exactly k zeros on $(-1, 1)$.*

(c) *$V_n(t)$ is positive on $(-1, 1)$ and strictly increasing on $[-1, 0]$.*

Proof. (a) This follows from (1.3), since (2.3) and (2.4) imply

$$\int_{-1}^{-t} (-t-u)^{n-1}\, sg_n(u)\, du = \int_{t}^{1} (-t+v)^{n-1}\, sg_n(-v)\, dv$$

$$= -\int_{t}^{1} (t-v)^{n-1}\, sg_n(v)\, dv$$

$$= \int_{-1}^{t} (t-v)^{n-1}\, sg_n(v)\, dv.$$

(b) Equation (3.1) and (a) imply that $+1$ and -1 are zeros of $V_n(t)$ of order $(n-1)$. Using Rolle's theorem k times, we deduce that $V_n^{(k)}(t)$ has at least k zeros on $(-1, 1)$. Suppose now that $V_n^{(k)}(t)$ has more than k zeros on $(-1, 1)$. Then (again by Rolle's theorem) $V_n^{(n-1)}(t)$ possesses at least n zeros on $(-1, 1)$. But $V_n^{(n-1)}(t)$ is a piecewise linear function, that vanishes for $t=\pm 1$ and which consists of $(n+1)$ linear (and nonconstant) pieces; therefore $V_n^{(n-1)}(t)$ has at most $(n-1)$ zeros on $(-1, 1)$.

(c) By (2.3) we have $sg_n(t)=1$ for $t<x_1$, and therefore (using (1.3)) $V_n(t)>0$ for $-1<t<x_1$. Because of (b) we deduce $V_n(t)>0$ on $(-1, 1)$. Since $V_n(t)$ is an even function, the zero of $V_n'(t)$ is $x=0$. Since $V_n(-1)=0$ and $V_n(t)>0$ on $(-1, 1)$ we have $V_n'(t)>0$ on $(-1, 0)$.

Next we state the above-cited result of Markoff and some consequences of it.

THEOREM 2 (Representations and estimates of $E_{n-1}(f)$).

(a) $E_{n-1}(f) \geqslant |\int_{-1}^{1} f(t)\, sg_n(t)\, dt|$ *if* $f \in L^1(I)$.

(b) (*Markoff*) $E_{n-1}(f) = \|f - p_f\|_1 = |\int_{-1}^{1} f(t)\, sg_n(t)\, dt|$ *if* $f \in M_n$.

(c) $E_{n-1}(f) = |\int_{-1}^{1} V_n(t)\, dg(t)|$ *if* $f \in M_n \cap F_n$.

(d) $|\int_{-1}^{1} V_n(t)\, dg(t)| \leqslant E_{n-1}(f) \leqslant \int_{-1}^{1} V_n(t)\, |dg(t)|$ *if* $f \in F_n$.

Proof. (a) $E_{n-1}(f) = \|f - p^*\|_1 \geqslant |\int_{-1}^{1} (f - p^*)\, sg_n| = |\int_{-1}^{1} f sg_n|$ with $p^* \in P_{n-1}$ and the use of (2.4).

(b) See [2, p. 91].

(c) If $\varphi(t)$ is integrable and $g(t)$ is of bounded variation on I, then

$$\int_{-1}^{1} dx\, \varphi(x) \int_{-1}^{x} (x-t)^{n-1}\, dg(t) = \int_{-1}^{1} dg(t) \int_{t}^{1} (x-t)^{n-1}\, \varphi(x)\, dx, \quad (3.2)$$

and therefore (use (2.13), (2.4) and (1.3))

$$\int_{-1}^{1} sg_n(x) f(x)\, dx = (-1)^n \int_{-1}^{1} V_n(t)\, dg(t) \qquad \text{for} \quad f \in F_n. \quad (3.3)$$

This and (b) imply the proposition.

(d) The left inequality is an immediate consequence of (a) and (3.3). Let $f(x) = p(x) + \int_{-1}^{x} ((x-t)^{n-1})/(n-1)!)\, dg(t)$. Put $h(x) = \int_{-1}^{x} ((x-t)^{n-1})/(n-1)!)\, |dg(t)|$. Then $h \pm f \in F_n^+ \subset M_n$, and therefore

$$\{(h - p_{h,n}) \pm (f - p_{f,n})\} \cdot \operatorname{sgn} U_n \geqslant 0.$$

We deduce

$$|f(x) - p_{f,n}(x)| \leqslant |h(x) - p_{h,n}(x)| \qquad \text{for} \quad -1 \leqslant x \leqslant 1,$$

and therefore

$$E_{n-1}(f) \leqslant \|f - p_f\|_1 \leqslant \|h - p_h\|_1 = E_{n-1}(h) = \int_{-1}^{1} V_n(t)\, |dg(t)|.$$

We continue with an alternative representation of $V_n(t)$.

PROPOSITION 1. *If* $n \geqslant 1$ *and* $-1 \leqslant t \leqslant 1$, *then*

$$n!\, V_n(t) = (t+1)^n + 2 \sum_{k=1}^{n} (-1)^k\, (t - x_k)_+^n.$$

Proof. By (2.2) and (2.3) we have

$$sg_n(u) = (1+u)_+^0 + 2 \sum_{k=1}^{n} (-1)^k (u - x_k)_+^0 \qquad \text{for} \quad u \neq x_0, x_1, \ldots, x_n,$$

and therefore (1.3) implies

$$(n-1)!\, V_n(t) = \int_{-1}^{t} (t-u)^{n-1}\, du + 2 \sum_{\substack{k=1 \\ x_k \leqslant t}}^{n} (-1)^k \int_{x_k}^{t} (t-u)^{n-1}\, du.$$

Remark. Since $(t - x_k)_+^n$ decreases in k, we have

$$n!\, V_n(t) \leqslant (1 - |t|)^n \qquad \text{for} \quad -1 \leqslant t \leqslant 1. \tag{3.4}$$

We conclude with some lower estimates of $V_n(t)$.

THEOREM 3 (Quantitative lower estimates of $V_n(t)$)

(a) $V_n(t) \geqslant \left(\dfrac{1-t^2}{2}\right)^n \cdot \dfrac{1}{n!}$,

(b) $V_n(t) \geqslant \dfrac{(n+1)\gamma_n}{\sqrt{2}} \left(\dfrac{1-t^2}{2}\right)^{n+1/2} \dfrac{1}{n!} = \dfrac{\pi}{4} V_n^*(t)$,

(c) $V_n(t) \geqslant \dfrac{1}{3} \tilde{V}_n(t)$.

Proof. (a) The nth Legendre-polynomial is given by [1, p. 334; #8.6.18] $P_n(x) = 1/(2^n n!)\,(d^n)/(dx^n)(x^2-1)^n$. Using $|P_n(x)| \leqslant 1$ for $-1 \leqslant x \leqslant 1$, and $\int_{-1}^{1} x^m P_n(x)\, dx = 0$ for $0 \leqslant m < n$, we obtain for any f with nonnegative continuous $f^{(n)}$:

$$\int_{-1}^{1} V_n(t) f^{(n)}(t)\, dt = E_{n-1}(f) = \int_{-1}^{1} |f - p_f| \geqslant \left| \int_{-1}^{1} (f - p_f) \cdot P_n \right|$$
$$= \left| \int_{-1}^{1} f \cdot P_n \right| = \int_{-1}^{1} f^{(n)}(x) \cdot \frac{(1-x^2)^n}{2^n \cdot n!}\, dx.$$

Since $f^{(n)}$ is an arbitrary nonnegative continuous function, the proposition follows.

(b) We have [1, p. 785; #22.11.4]

$$\sqrt{(1-x^2)}\, U_n(x) = \frac{(n+1)\sqrt{\pi}\,(-1)^n}{2^{n+1}\Gamma(n+3/2)} \frac{d^n}{dx^n} (1-x^2)^{n+1/2}$$
$$= \frac{(n+1)(-1)^n \gamma_n}{n!\, 2^{n+1}} \frac{d^n}{dx^n} (1-x^2)^{n+1/2}.$$

Using $\int_{-1}^{1} \sqrt{1-x^2}\, U_n(x)\, x^m\, dx = 0$ for $m < n$ [1, p. 774; #22.2.5], and

$$|U_n(x)\sqrt{1-x^2}| = |\sin(n+1)\,\Theta| \leqslant 1 \qquad \text{for} \quad -1 \leqslant x \leqslant 1,$$

we deduce for any f with nonnegative continuous $f^{(n)}$:

$$\int_{-1}^{1} V_n(t) f^{(n)}(t)\, dt \geqslant \left| \int_{-1}^{1} f(x)\, U_n(x) \sqrt{1-x^2}\, dx \right|$$
$$= \frac{(n+1)\,\gamma_n}{n!\, 2^{n+1}} \int_{-1}^{1} (1-x^2)^{n+1/2} f^{(n)}(x)\, dx.$$

(c) Add the inequalities (a) and (b) and use (2.9).

4. An Integral Representation of $V_n(t)$

THEOREM 4. *For $-1 < t < 1$ and $n \in \mathbb{N}$ we have*

$$V_n(t) = \frac{2\sqrt{2}(n+1)}{\pi n!} \left(\frac{1-t^2}{2}\right)^{n+1/2} \cdot \int_{-1}^{1} \frac{(1-u^2)^n\, du}{1-(t+iu\sqrt{1-t^2})^{2(n+1)}}.$$

Proof. Let $x_l < t < x_{l+1}$. Proposition 1 yields

$$n!\, V_n(t) = (t-x_0)^n + 2 \sum_{k=1}^{l} (-1)^k (t-x_k)^n$$
$$= (n+1) \cdot 2^{1-n} \sum_{\operatorname{Re} z < t} \operatorname{Res}_F(z), \tag{4.1}$$

since the rational function

$$F(z) = \frac{(z^2 - 2zt + 1)^n}{1 - z^{2(n+1)}} = \frac{z^n (z + 1/z - 2t)^n}{1 - z^{2(n+1)}}$$

has poles exactly at the $2(n+1)$th roots of unity and since (put $z_k = -\exp(k\pi i/(n+1))$ the corresponding residues are given by

$$\operatorname{Res}_F(z_k) = \frac{z_k^n \cdot 2^n (x_k - t)^n}{-2(n+1)\, z_k^{2n+1}} = \frac{-z_k^{n+1} 2^{n-1} (x_k - t)^n}{n+1}$$
$$= \frac{(-1)^k\, 2^{n-1} (t - x_k)^n}{n+1} = \operatorname{Res}_F(\bar{z}_k).$$

Let Γ_1, Γ_2, and Γ_3 be paths that join $t - i\sqrt{1-t^2}$ with $t + i\sqrt{1-t^2}$, where Γ_1 is a straightline, Γ_2 lies outside the unitcicle, -1 is positively encircled once by $\Gamma_1 - \Gamma_2$, and Γ_3 is the image of $-\Gamma_2$ under the mapping $z \mapsto 1/z$.

Since $F(1/z) = -z^2 F(z)$, the substitution $z = 1/u$ implies

$$\int_{-\Gamma_2} F(z)\,dz = \int_{\Gamma_3} F(z)\,dz, \tag{4.2}$$

and therefore (4.1), the residue theorem, (4.2), and Cauchy's theorem yield

$$n!\,V_n(t) = \frac{(n+1)\,2^{-n}}{\pi i} \int_{\Gamma_1 - \Gamma_2} F(z)\,dz = \frac{(n+1)\,2^{1-n}}{\pi i} \int_{\Gamma_1} F(z)\,dz. \tag{4.3}$$

This implies the desired result by putting $z = z(u) = t + iu\sqrt{1-t^2}$. By continuity we get the desired result for $t \in \{x_k \mid 0 < k \leqslant n\}$ too.

Since $(z^2 - 2zt + 1)$ vanishes at the endpoints of Γ_1, we obtain by differentiating (4.3) with respect to t for $0 \leqslant k < n$:

$$\begin{aligned}
&(n-k)!\,V_n^{(k)}(t)\\
&= \frac{(n+1)(-1)^k}{\pi i}\,2^{1+k-n} \int_{\Gamma_1} \frac{(z^2 - 2zt + 1)^{n-k} \cdot z^k}{1 - z^{2(n+1)}}\,dz\\
&= \frac{2\sqrt{2}\,(n+1)(-1)^k}{\pi} \left(\frac{1-t^2}{2}\right)^{n+1/2-k} \int_{-1}^{1} \frac{(1-u^2)^{n-k}\,(t + iu\sqrt{1-t^2})^k}{1 - \{t + iu\sqrt{1-t^2}\}^{2(n+1)}}\,du.
\end{aligned}$$

For $t = 0$ and $k = 2m < n$ we obtain, by comparing with Theorem 3:

$$\begin{aligned}
&\sum_{l=-[n/2]}^{[n/2]} (-1)^l \cdot \left\{\cos\frac{l\pi}{n+1}\right\}^{n-2m}\\
&\quad = (-1)^m \frac{(n+1)\,2^{2m+1-n}}{\pi} \cdot \int_{-1}^{1} \frac{(1-u^2)^{n-2m}\,u^{2m}\,du}{1 + (-1)^n\,u^{2(n+1)}}
\end{aligned}$$

and especially (for $m = 0$) (1.7).

We remark that Gould [5] proved

$$\sum_{l=-[n/2]}^{[n/2]} (-1)^l \left(\cos\frac{\pi l}{n+1}\right)^{n+1} = (n+1)\,2^{-n}$$

an identity that is equivalent to $\int_{-1}^{0} V_n(t)\,dt = 2^{-n}$.

For $-1 < t < 1$ and $0 < \varphi < \pi$ the partial sums of $\sum_{j=0}^{\infty} \{t + i\sqrt{1-t^2}\cos\varphi\}^{2j(n+1)}$ are dominated by $2/((1-t^2)\sin^2\varphi)$. Using in Theorem 4 the substitution $u = \cos\varphi$, we obtain

$$V_n(t) = \frac{2(n+1)}{\pi \cdot 2^n} \frac{(1-t^2)^{n+1/2}}{n!} \int_0^\pi \frac{(\sin\varphi)^{2n+1}}{1-(t-\sqrt{t^2-1}\cos\varphi)^{2(n+1)}}\, d\varphi$$

$$= \frac{2(n+1)}{\pi \cdot 2^n n!} (1-t^2)^{n+1/2}$$

$$\cdot \sum_{j=0}^{\infty} \int_0^\pi (\sin\varphi)^{2n+1} \{t+\sqrt{t^2-1}\cos\varphi\}^{2j(n+1)}\, d\varphi,$$

and therefore [1, p. 784; #22.10.10, and p. 777; #22.4.2] the following expansion in Gegenbauer polynomials.

PROPOSITION 2.

$$V_n(t) = \frac{2^{n+2}}{\pi \binom{2n+1}{n}} \frac{(1-t^2)^{n+1/2}}{n!} \sum_{j=0}^{\infty} \frac{C_{2j(n+1)}^{(n+1)}(t)}{C_{2j(n+1)}^{(n+1)}(1)}$$

holds for $-1 < t < 1$ *and* $n \in \mathbb{N}$.

5. THE MAGNITUDE OF $V_n(t)$

Put ($n \geqslant 1$; $-1 < t < 1$)

$$V_n(t) = V_n^*(t) \cdot \{1 + R_n(t)\} \tag{5.1}$$

with

$$\gamma_n \cdot R_n(t) = \int_{-1}^{1} \frac{(1-u^2)^n \{t+iu\sqrt{1-t^2}\}^{2(n+1)}\, du}{1-\{t+iu\sqrt{1-t^2}\}^{2(n+1)}},$$

and therefore

$$\gamma_n \cdot |R_n(t)| \leqslant \int_{-1}^{1} \frac{(1-u^2)^n \{t^2+u^2(1-t^2)\}^{n+1}\, du}{1-\{t^2+u^2(1-t^2)\}^{n+1}}. \tag{5.2}$$

LEMMA 1. ($n \geqslant 1$; $-1 < t < 1$.)

(a) $|R_n(t)| \leqslant 1/n(1-t^2)$,

(b) $\gamma_n \cdot |R_n(t)| \leqslant 8 \cdot \{4(1-t^2)\}^{-n}$, *for* $t^2 \leqslant \frac{1}{2}$

$\leqslant 2t^{2(n-1)}/(1-t^2)$, *for* $t^2 \geqslant \frac{1}{2}$.

Proof. (a) Since $0 \leqslant b < 1$ implies $1 - b^{n+1} \geqslant (n+1)(1-b)\, b^n$, we deduce from (5.2):

$$\gamma_n\,|R_n(t)| \leqslant \frac{1}{(n+1)(1-t^2)} \int_{-1}^{1} (1-u^2)^{n-1}\, \{t^2 + u^2(1-t^2)\}\, du$$

$$\leqslant \frac{1}{(n+1)(1-t^2)}\, \gamma_{n-1} = \frac{2n+1}{2n(n+1)(1-t^2)} \cdot \gamma_n.$$

(b) Using

$$\frac{1-u^2}{1 - \{t^2 + u^2(1-t^2)\}^{n+1}} \leqslant \frac{1-u^2}{1 - \{t^2 + u^2(1-t^2)\}} = \frac{1}{1-t^2},$$

we obtain from (5.2):

$$\gamma_n\,|R_n(t)| \leqslant \frac{1}{1-t^2} \int_{-1}^{1} \{(1-u^2)(t^2 + u^2[1-t^2])\}^{n-1}\, du. \tag{5.3}$$

This gives the desired result by estimating the integrand in (5.3) by its maximal value; this integrand is maximal for $1 - u^2 = 1/(2(1-t^2))$ if $t^2 \leqslant \frac{1}{2}$, and for $u = 0$ if $t^2 \geqslant \frac{1}{2}$.

THEOREM 5. $(n \geqslant 1; -1 < t < 1.)$

(a) $\frac{1}{3}\, \tilde{V}_n(t) \leqslant V_n(t) \leqslant 3\tilde{V}_n(t)$.

(b) $|V_n(t) - V_n^*(t)/ \leqslant \dfrac{2}{\pi} \sqrt{1-t^2}\, \dfrac{(n+1)}{n!}\, \varphi^{n-1}(t)$,

with

$$\varphi(t) = \tfrac{1}{8}, \qquad \textit{for} \quad t^2 \leqslant \tfrac{1}{2}$$
$$= t^2(1-t^2)/2, \qquad \textit{for} \quad t^2 \geqslant \tfrac{1}{2}.$$

Proof. (a) The left inequality was stated in Theorem 3(c). Let $t^2 < 1 - 1/n$ and thus $n \geqslant 2$. Then (5.1), Lemma 1a, (1.5), (1.6), and (2.7) yield (observe $V_n^*(t)/\tilde{V}_n(t) \searrow$ in t^2) $V_n(t) \leqslant 2V_n^*(t) \leqslant 3\tilde{V}_n(t)$. Let now $t^2 \geqslant 1 - 1/n$, and thus $n(1-|t|)/(1+|t|) \leqslant (1+|t|)^{-2} \leqslant 1$. Using (3.4) we obtain

$$V_n(t) = V_n(-|t|) \leqslant \frac{(1-|t|)^n}{n!} = \left(\frac{1-t^2}{2}\right)^n \left(\frac{2}{1+|t|}\right)^n \frac{1}{n!},$$

$$\leqslant \tilde{V}_n(t) \cdot \left(1 + \frac{1-|t|}{1+|t|}\right)^n \leqslant \tilde{V}_n(t) \exp\left(n\, \frac{1-|t|}{1+|t|}\right) \leqslant 3\tilde{V}_n(t).$$

(b) This follows from (5.1), (1.6) and Lemma 1(b).

6. The Asymptotic Behaviour of $V_n(t)$

In this section we will determine a rather simple function $*V_n(t)$ with $V_n(t) = *V_n(t) \cdot \{1 + O(1/n)\}$, where $O(\cdot)$ is independent of n and t. We start with a simple consequence of Lemma 1 (use (2.7)).

LEMMA 2. *$V_n(t) = V_n^*(t) \cdot \{1 + R_n(t)\}$ with*

$$|R_n(t)| \leqslant \frac{3}{n} \qquad \textit{for} \quad |t| \leqslant 1 - \frac{3 \log n}{2n} \ \textit{and} \ n \geqslant 2.$$

Remark. Similarly one obtains $|R_n(t)| \leqslant c_k \cdot n^{-k}$ for $|t| \leqslant 1 - (k+1) \times \log n/n$, and even $R_n(t) = O(\sqrt{n}\, e^{-\sqrt{n}})$ for $|t| \leqslant 1 - 1/\sqrt{n}$. Next we will show that $(1 - |t|)^n/n!$ is a good approximation for $V_n(t)$ in the neighbourhood of 1.

LEMMA 3.

$$V_n(t) = \frac{(1 - |t|)^n}{n!} (1 + r_n(t))$$

with

$$|r_n(t)| \leqslant \frac{1}{n} \qquad \textit{for} \quad 1 > |t| \geqslant 1 - \frac{1}{(n+1) \log n} \ \textit{and} \ n \geqslant 2.$$

Proof. Since $V_n(t)$ is an even function, we may suppose $t \leqslant 0$. $(1 - \cos x) \cdot x^{-2}$ decreases on $[0, \pi/2]$, and therefore

$$(2/\pi)^2 \leqslant (1 - \cos x)\, x^{-2} \leqslant \tfrac{1}{2} \qquad \text{for} \quad |x| \leqslant \pi/2. \tag{6.1}$$

Proposition 1 represents $V_n(t)$ by an alternating series and implies

$$|r_n(t)| = \left| \frac{n!\, V_n(t)}{(1 - |t|)^n} - 1 \right| \leqslant 2 \left(\frac{t - x_1}{1 + t} \right)_+^n \qquad \text{for} \quad -1 < t \leqslant 0,$$

and by (6.1) we have (for $x_1 < t \leqslant -1 + 1/((n+1) \log n)$)

$$\left(\frac{t - x_1}{1 + t} \right)^n = \left(1 - \frac{1 + x_1}{1 + t} \right)^n \leqslant \exp \left\{ -n \frac{1 + x_1}{1 + t} \right\}$$

$$\leqslant \exp \left\{ \frac{-4n}{(n+1)^2 (1 + t)} \right\} \leqslant \frac{1}{n^2}.$$

Remark. Similarly one obtains $r_n(t) \leqslant n^{-k}$ for $|t| \geqslant 1 - 1/(k(n+1) \log n)$ and $n \geqslant 2$ and even $|r_n(t)| \leqslant e^{-\sqrt{n}}$ for $|t| \geqslant 1 - (n+1)^{-3/2}$.

THEOREM 6. *For $n \geqslant n_0$ and $|t| < 1$ we have*

$$V_n(t) = {}^*V_n(t) \cdot \{1 + O(1/n)\}$$

with

$$\begin{aligned} {}^*V_n(t) &= V_n^*(t) && \text{if } |t| \leqslant 1 - \frac{3}{2}\frac{\log n}{n}, \\ &= \frac{(1-|t|)^n}{n!} && \text{if } |t| \geqslant 1 - \frac{1}{(n+1)\log n}, \\ &= \left(\frac{1-t^2}{2}\right)^n \frac{1}{n!} e^{\sigma/4} \left\{F(\sigma) + \frac{G(\sigma)}{n+1}\right\} && \text{else,} \end{aligned}$$

where

$$\sigma = (n+1)(1-t^2),$$

and

$$G(\sigma) = \left(\frac{3\sigma^2 - 8\sigma}{32}\right) F(\sigma) + \frac{\sigma^2}{4} F'(\sigma) - \frac{\sigma^2}{2} F''(\sigma).$$

Proof. It remains the case when

$$t \in \left[-1 + \frac{1}{(n+1)\log n}, -1 + \frac{3\log n}{2n}\right],$$

which implies (for $n \geqslant 3$)

$$1 - \frac{3\log n}{n+1} \leqslant t^2 \leqslant 1 - \frac{1}{(n+1)\log n}. \tag{6.2}$$

We write $V_n(t) = 1/(n!)((1-t^2)/2)^n \cdot F_n(\sigma)$ with (according to Proposition 1)

$$F_n(\sigma) = \left(\frac{2(t-x_0)}{1-t^2}\right)^n + 2\sum_{k=1}^{n} (-1)^k \left(\frac{2(t-x_k)_+}{1-t^2}\right)^n. \tag{6.3}$$

We have to handle terms of the form

$$\left(\frac{2(t-x_k)}{1-t^2}\right)^n = (1-\eta)^n$$

with

$$\eta = \eta_k = \frac{2(1-\cos w_k)}{z} - \frac{2 - z - 2\sqrt{1-z}}{z},$$

where

$$z = \frac{\sigma}{n+1} = 1 - t^2 \qquad \text{and} \qquad w_k = \frac{\pi k}{n+1}.$$

Because of (6.2) we may restrict our considerations to $1/\log n \leqslant \sigma \leqslant 3 \log n$. Furthermore, we suppose $n \geqslant n_0$. By (6.1) we have

$$0 < z < \tfrac{1}{4}, \qquad \text{and} \qquad \eta_k \geqslant (w_k^2/z - z)/2. \tag{6.4}$$

Put $l := [2\sqrt{\sigma \log n}] + 1$. We have by (6.4)

$$x_{l-1} \leqslant t, \qquad \text{and} \qquad w_l/z \geqslant \pi,$$

and therefore (6.4) implies $z \cdot \eta_l \geqslant w_l^2/3$, and

$$(1 - \eta_l)_+^n \leqslant \exp(-n\eta_l) \leqslant \exp(-nw_l^2/(3z)) \leqslant n^{-4}. \tag{6.5}$$

Now we consider $0 \leqslant k < l$. We have

$$w_k^2/z < \tfrac{1}{4} \qquad \text{and} \qquad |\eta_k| \leqslant \max(w_k^2/z, z/2) < \tfrac{1}{4},$$

and

$$\eta_k = w_k^2/z - z/4 - z^2/8 + O(w_k^4/z + z^3).$$

We deduce

$$\log(1 - \eta_k) = \left\{\frac{z}{4} - w_k^2 z^{-1} + \left\{\frac{3}{32} z^2 + \frac{1}{4} w_k^2 - \frac{1}{2} w_k^4 z^{-2}\right\}\right.$$
$$\left. + O(z^3 + w_k^6 z^{-3}),\right.$$

and (write $n \log(1 - \eta_k)) = (n+1)\log(\cdots) - n \log(\cdots))$

$$(1 - \eta_k)^n = e^{\sigma/4} e^{-k^2\pi^2/\sigma} \left\langle 1 + \frac{1}{n+1} \left\{ \frac{3\sigma^2 - 8\sigma}{32} + k^2\pi^2 \left(\frac{1}{4} + \frac{1}{\sigma}\right)\right.\right.$$
$$\left.\left. - \frac{k^4\pi^4}{2\sigma^2} \right\} + O\left(\frac{\log^4 n}{n^2}\right)\right\rangle,$$

and

$$e^{-\sigma/4} \cdot \left\{(1 - \eta_0)^n + 2 \sum_{k=1}^{l-1} (-1)^k (1 - \eta_k)^n \right\}$$
$$= \sum_{|k| < l} (-1)^k e^{-k^2\pi^2/\sigma} \left\langle 1 + \frac{1}{(n+1)} \left\{ \frac{3\sigma^2 - 8\sigma}{32} \right.\right.$$
$$\left.\left. + k^2\pi^2 \left(\frac{1}{4} + \frac{1}{\sigma}\right) - \frac{k^4\pi^4}{2\sigma^2} \right\} \right\rangle + O(n^{-2}\log^4 n) \sum_{k=-\infty}^{\infty} e^{-k^2\pi^2/\sigma}$$

$$= F(\sigma) + \frac{1}{n+1} \left\{ \left(\frac{3\sigma^2 - 8\sigma}{32} \right) F(\sigma) \right.$$

$$\left. + \sum_{k=-\infty}^{\infty} (-1)^k e^{-k^2\pi^2/\sigma} \left\langle \left(\frac{\sigma^2}{4} + \sigma \right) \frac{k^2\pi^2}{\sigma^2} - \frac{k^4\pi^4}{2\sigma^2} \right\rangle \right\}$$

$$+ O(\log^5 n/n^2),$$

since $\sum_{k=-\infty}^{\infty} e^{-k^2\pi^2/\sigma} = O(1 + \sqrt{\sigma})$ and (for example)

$$\left| \sum_{k=l}^{\infty} (-1)^k e^{k^2\pi^2/\sigma} \right| \leqslant e^{-l^2\pi^2/\sigma} \leqslant e^{-4\pi^2 \log n} \leqslant n^{-2}.$$

Using (6.5) and

$$F'(\sigma) = \sum_{k=-\infty}^{\infty} (-1)^k k^2\pi^2\sigma^{-2} e^{-k^2\pi^2/\sigma}$$

and

$$(\sigma^2 F'(\sigma))' = \sum_{k=-\infty}^{\infty} (-1)^k k^4\pi^4\sigma^{-2} e^{-k^2\pi^2/\sigma},$$

we obtain

$$F_n(\sigma) = e^{\sigma/4}(F(\sigma) + G(\sigma)/(n+1) + O(n^{-2} \log^5 n)).$$

This implies the assertion, since we have $F(\sigma) \geqslant n^{-3/4}$ by (2.9), and $G(\sigma) = O(\log^2 n)$ by $\sigma^k F^{(k)}(\sigma) = O(1)$ for $k = 1, 2$ (note that $\sigma F'(\sigma)$ and $(\sigma^2 F'(\sigma))'$ are alternating series with bounded terms).

Remark. From (2.8) and (2.9) one may obtain

$$F'(\sigma) = F(\sigma) \cdot O(1 + \sigma^{-1}), \qquad \text{and} \qquad F''(\sigma) = F(\sigma) \cdot O(1 + \sigma^{-2}).$$

This yields $G(\sigma) = F(\sigma) \cdot O(1 + \sigma^2)$, and thus

$$\left(F(\sigma) + \frac{G(\sigma)}{n+1} \right) (1 + O(1/n)) = F(\sigma) \cdot (1 + O((1 + \sigma^2)/n))$$

$$= F(\sigma) \cdot (1 + O(n^{-1} \log^2 n)).$$

7. Examples

If f is continuous, then, according to a theorem of Jackson [9, p. 69], the minimizing polynomial is uniquely determined. Therefore, the minimizing polynomial of a continuous function f is even if f is even, and odd if f is odd. This result and [2, p. 94] $E_{n-1}(x^n) = 2^{1-n}$ imply

$$E_{n-2}(x^n) = E_{n-1}(x^n) = 2^{1-n}. \tag{7.1}$$

LEMMA 4. *$g(x^2) \in M_{2n}$ and $xg(x^2) \in M_{2n+1}$ if $g^{(n)}(x) \neq 0$ on $[0, 1)$.*

Proof. Put $f(x) = g(x^2)$. We shall show that $f - p_{f,2n-1}$ has exactly the zeros $x_{2n,1}; x_{2n,2}; \cdots; x_{2n,2n}$, and changes sign at these points. Note that $x_{2n,k} \neq 0$. Since f is even, $p_{f,2n-1}$ is even too. If

$$f - p_{f,2n-1} = g(x^2) - \sum_{k=0}^{n-1} a_k x^{2k} = g(x^2) - p_{n-1}(x^2)$$

had another zero or a multiple zero on $(-1, 1)$, then $g(u) - p_{n-1}(u)$ would have at least $(n+1)$ zeros on $[0, 1)$. This is impossible by Rolle's theorem. The function $xg(x^2)$ may be handled similarly.

Next we collect some identities and inequalities.

LEMMA 5 (k denotes a nonnegative integer.)

(a) $$\int_{-1}^{1} V_n(t)\, dt = 2^{1-n}/n!.$$

(b) $$\int_{-1}^{1} V_n^*(t) \cdot t^{2k}\, dt = 2^{1-n-2k} \frac{(2k)!\,(n+1)}{k!(n+k+1)!}.$$

(c) $$\int_{-1}^{1} \tilde{V}_n(t) \cdot t^{2k}\, dt = \pi \cdot 2^{-n-2k} \frac{(2k)!}{k!(n+k+1)!} \times \left\{ \frac{\sqrt{n+1}}{\gamma_n} + \frac{1}{\gamma_{n+k+1/2}} \right\}.$$

(d) $$\int_{-1}^{1} |V_n(t) - V_n^*(t)|\, t^{2k}\, dt \leqslant \frac{8^{1-n}}{n!} \frac{n+1}{k+1}.$$

Proof. (a) Apply Theorem 2(c) to the function $f(x) = x^n$ and use (7.1).

(b) Using (1.6), (2.5) and (for $\alpha, \beta \geqslant 0$)

$$\int_{-1}^{1} (1-t^2)^\alpha\, t^{2\beta}\, dt = B\left(\alpha+1, \beta+\frac{1}{2}\right) = \frac{\Gamma(\alpha+1)\,\Gamma(\beta+\frac{1}{2})}{\Gamma(\alpha+\beta+\frac{3}{2})}, \tag{7.2}$$

we obtain

$$\int_{-1}^{1} V_n^*(t) \cdot t^{2k}\, dt = \frac{2^{1-n}(n+1)\,\Gamma(\frac{1}{2})\,\Gamma(n+1)\,\Gamma(n+\frac{3}{2}\,\Gamma(k+\frac{1}{2})}{\pi \cdot n!\,\Gamma(n+\frac{3}{2}) \cdot \Gamma(n+k+2)}$$

$$= \frac{2^{1-n}(n+1)\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\pi(n+k+1)!} \prod_{i=1}^{k} \left(\frac{2j-1}{2}\right).$$

(c) This is proven similarly to (b).

(d) This follows from Theorem 5(b) and from

$$\int_{-1}^{1} \sqrt{1-t^2} \cdot t^{2k}\, dt = B\left(\frac{3}{2}, k+\frac{1}{2}\right) = \frac{\Gamma(\frac{3}{2})\,\Gamma(k+\frac{1}{2})}{\Gamma(k+2)}.$$

For $n \geqslant 1$ and $k \geqslant 0$ we put

$$c_{n,k} = \frac{2^{n-1} \cdot (n+2k)!}{(2k)!} \cdot \int_{-1}^{1} V_n(t) \cdot t^{2k}\, dt.$$

We have $c_{n,0} = 1$ by Lemma 5. Theorem 5, Lemma 5, and (2.7) imply

$$c_{n,k} \leqslant 8\,\frac{(n+2k)!}{4^k k!(n+k)!} = 8 \cdot 4^{-k} \cdot \binom{n+2k}{k}, \tag{7.3}$$

and

$$\frac{1}{8}\binom{n+2k}{k}\frac{n+1+\sqrt{k}}{n+1+k} \leqslant 4^k \cdot c_{n,k} \leqslant 8\binom{n+2k}{k}\frac{(n+1+\sqrt{k})}{n+k+1}. \tag{7.4}$$

Furthermore, we have by Lemma 5 ($\varepsilon = \varepsilon_{n,k}$ with $|\varepsilon| \leqslant 1$)

$$c_{n,k} = \frac{(n+1)(n+2k)!}{4^k \cdot k!(n+k+1)!}\left\{1 + \varepsilon \cdot \frac{4^{k+1-n}k!(n+k+1)!}{(k+1)(2k)!\,n!}\right\}.$$

Since $(2^k k!(n+k+1)!)/((k+1)(2k)!)$ increases for $k \leqslant n$ and decreases for $k \geqslant n$, we obtain

$$c_{n,k} = \frac{(n+1)(n+2k)!}{4^k \cdot k!(n+k+1)!}\{1 + \varepsilon \cdot 2^{3+k-n}\}. \tag{7.5}$$

THEOREM 7. (a) *If $f \in C^{n+2}(I)$, then*

$$2^{n-1} \cdot n!\, E_{n-1}(f) = |f^{(n)}(0)| + O(1/n)\{|f^{(n+1)}(0)| + d_{n,1}\},$$

where

$$d_{n,1} = \sup_I |f^{(n+2)}(t)| \, (1-t^2)^{n/2}.$$

(b) *If* $f \in M_n$ *possesses a Maclaurin expansin* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, *which is convergent for* $|z| \leqslant 1$, *then*

$$2^{n-1} \cdot E_{n-1}(f) = \left| \sum_{k=0}^{\infty} c_{n,k} a_{n+2k} \right|.$$

Proof. (a) We have $f^{(n)}(t) = f^{(n)}(0) + tf^{(n+1)}(0) + h(t)$ and

$$\begin{aligned}
\int_{-1}^{1} V_n(t) \, |h(t)| \, dt &= \int_{-1}^{1} V_n(t) \left| \int_0^t (t-u) f^{(n+2)}(u) \, du \right| dt, \\
&\leqslant \int_{-1}^{1} V_n(t) \cdot t \int_0^t |f^{(n+2)}(u)| \, du \, dt, \\
&\leqslant d_{n,1} \cdot \int_0^1 \tilde{V}_n(t) \cdot t \cdot \int_0^t (1-u^2)^{-n/2} \, du,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \tilde{V}_n(t) \cdot t \cdot \int_0^t (1-u^2)^{-n/2} \, du \\
&\quad \leqslant \frac{2^{-n}}{n!} \int_0^1 (1-u^2)^{-n/2} \int_u^1 t(1-t^2)^n \cdot \left\{ 1 + \sqrt{(n+1)(1-t^2)} \right\} dt \, du \\
&\quad = \frac{2^{-n}}{n!} \left\{ \frac{1}{2(n+1)} \cdot \gamma_{1+n/2} + \frac{\sqrt{n+1}}{2n+3} \cdot \gamma_{(n+3)/2} \right\} \\
&\quad = \frac{2^{1-n}}{n!} \, O(1/n).
\end{aligned}$$

Using Theorem 2 and the triangle-inequality, and Lemma 5(a), we obtain

$$\begin{aligned}
E_{n-1}(f) &\geqslant \left| \int_{-1}^{1} V_n(t) \{ f^{(n)}(0) + tf^{(n+1)}(0) \} \, dt \right| - \int_{-1}^{1} V_n(t) \, |h(t)| \, dt, \\
&\geqslant \frac{2^{1-n}}{n!} \left\{ |f^{(n)}(0)| - O\left(\frac{d_{n,1}}{n}\right) \right\}.
\end{aligned}$$

On the other hand, we have (using (7.1) and Theorem 2(d))

$$E_{n-1}(f) \leqslant E_{n-1}\left(\frac{f^{(n)}(0)\, t^n}{n!}\right) + E_{n-1}\left(\frac{f^{(n+1)}(0)\, t^{n+1}}{(n+1)!}\right)$$
$$+ E_{n-1}\left(f(t) - \frac{f^{(n)}(0)\, t^n}{n!} - \frac{f^{(n+1)}(0)}{(n+1)!}\, t^{n+1}\right),$$
$$\leqslant \frac{2^{1-n}}{n!}\, |f^{(n)}(0)| + \frac{2^{-n}}{(n+1)!}\, |f^{(n+1)}(0)| + \int_{-1}^{1} V_n(t)\, |h(t)|\, dt$$
$$= \frac{2^{1-n}}{n!}\, \{|f^{(n)}(0)| + O(1/n)(|f^{(n+1)}(0)| + d_{n,1})\}.$$

(b) $$\pm E_{n-1}(f) = \int_{-1}^{1} V_n(t) f^{(n)}(t)\, dt$$
$$= \sum_{k=0}^{\infty} a_{n+k} \frac{(n+k)!}{k!} \int_{-1}^{1} V_n(t)\, t^k\, dt,$$
$$= 2^{1-n} \sum_{l=0}^{\infty} a_{n+2l} c_{n,l}.$$

In general we need $f^{(n)}(t) \neq 0$ on I in order to apply Theorem 7(b). If $f^{(n)}(t)$ changes sign on I, one often may use the following theorem (put $d_{n,k} = \sup_I |f^{(n+2k)}(t)| \cdot (1-t^2)^{n/2}$ for $n \geqslant 1$ and $k \geqslant 0$).

THEOREM 8 ($n \geqslant 1$; $k \geqslant 1$; $0 \leqslant \delta_n \leqslant 1$; $\gamma_{n,k} \geqslant 0$)

(a) *If $f \in C^{n+2k}(I)$ and $f^{(n)}(t) + \gamma_{n,k} t^{2k} \geqslant 0$ for $|t| \leqslant \delta_n$, then*

$$2^{n-1} \cdot n!\, E_{n-1}(f) = \sum_{m=0}^{k-1} c_{n,m} \frac{n!}{(n+2m)!} f^{(n+2m)}(0)$$
$$+ O(1)(1-\delta_n^2)^{n/4} \cdot d_{n,0} + O_k(n^{-k})(\gamma_{n,k} + d_{n,k}).$$

(b) *If f possesses a Maclaurin expansion,*

$$f(z) = \sum_{m=0}^{\infty} a_m z^m,$$

which is convergent for $|z| \leqslant 1$, and if $f^{(n)}(t) \geqslant 0$ for $|t| \leqslant \delta_n$, then

$$2^{n-1} E_{n-1}(f) = \left| \sum_{m=0}^{\infty} c_{n,m} a_{n+2m} \right| + O(1)(1-\delta_n^2)^{n/4} \cdot d_{n,0}/n!.$$

Remark. The first error term in (a) stems from the fact that we do not know the sign of $f^{(n)}(t)$ for $|t| \geqslant \delta_n$. The second error term comes in since we replace the assumption $f^{(n)}(t) \geqslant 0$ by $f^{(n)}(t) + \gamma_{n,k} t^{2k} \geqslant 0$ for $|t| \leqslant \delta_n$. The last error term finally is a consequence of the truncation of the Taylor-series expansion of $f^{(n)}(t)$. The last two error terms may be written as

$$O(n^{-k}) \left(\frac{(2k)!}{4^k k!} \cdot \gamma_{n,k} + \frac{2^k \Gamma(k+\frac{1}{2})}{\Gamma(2k)} d_{n,k} \right).$$

Proof (of Theorem 8). (a) We have

$$\begin{aligned} |f^{(n)}(t)| &\leqslant f^{(n)}(t) + 2\gamma_{n,k} t^{2k}, \qquad \text{for} \quad |t| \leqslant \delta_n, \\ &\leqslant d_{n,0} \cdot (1-t^2)^{-n/2}, \qquad \text{else.} \end{aligned} \tag{7.6}$$

By Theorem 2(d), (7.6) and Theorem 5, and (7.4) we obtain

$$\begin{aligned} 0 &\leqslant E_{n-1}(f) - \int_{-1}^{1} V_n \cdot f^{(n)} \leqslant \int_{-1}^{1} V_n \cdot \{|f^{(n)}| - f^{(n)}\} \\ &\leqslant \gamma_{n,k} \cdot \int_{-1}^{1} V_n(t) \cdot t^{2k}\, dt + d_{n,0} \int_{\delta_n}^{1} \tilde{V}_n(t)(1-t^2)^{-n/2}\, dt \\ &= \frac{2^{1-n}}{n!} \left\{ O(n^{-k}) \cdot \frac{(2k)!}{4^k k!} \gamma_{n,k} + O((1-\delta_n^2)^{n/4})\, d_{n,0} \right\}, \end{aligned}$$

since (by (1.5), (2.5) and (2.7))

$$\begin{aligned} &2^{n-1} \cdot n! \int_{\delta_n}^{1} \tilde{V}_n(t)(1-t^2)^{-n/2}\, dt \\ &\qquad \leqslant \sqrt{n} \int_{\delta_n}^{1} (1-t^2)^{n/2}\, dt \leqslant \sqrt{n}\,(1-\delta_n^2)^{n/4} \cdot \gamma_{n/4} \leqslant (1-\delta_n^2)^{n/4}. \end{aligned} \tag{7.7}$$

Using Mclaurin's expansion for $f^{(n)}(t)$ we get

$$\begin{aligned} &\left| \int_{-1}^{1} V_n(t) f^{(n)}(t)\, dt - 2^{1-n} \sum_{m=0}^{k-1} \frac{c_{n,m}}{(n+2m)!} f^{(n+2m)}(0) \right| \\ &\quad = \left| \int_{-1}^{1} V_n(t) \int_0^t \frac{(t-u)^{2k-1}}{(2k-1)!} f^{(n+2k)}(u)\, du\, dt \right|, \\ &\quad \leqslant \frac{d_{n,k}}{(2k-1)!} \int_{-1}^{1} \tilde{V}_n(t)\, t^{2k-1} \int_0^t (1-u^2)^{-n/2}\, du\, dt, \\ &\quad \leqslant \frac{d_{n,k}}{(2k-1)!} \int_{-1}^{1} \tilde{V}_n(t)\, t^{2k}(1-t^2)^{-n/2}\, dt, \\ &\quad \leqslant \frac{d_{n,k}}{(2k-1)!} \cdot \frac{2^{1-n}}{n!} 2^k \Gamma\left(k + \frac{1}{2}\right) n^{-k}, \end{aligned}$$

where the last inequality follows by (1.5), (7.2), (2.5), and (2.7).

(b) By Theorem 2(d) we have

$$E_{n-1}(f) \leqslant \int_{-1}^{1} V_n |f^{(n)}| = \int_{-\delta_n}^{\delta_n} V_n f^{(n)} + \int_{|t| \geqslant \delta_n} V_n |f^{(n)}|$$

$$= \int_{-1}^{1} V_n f^{(n)} + \int_{|t| \geqslant \delta_n} V_n \cdot \{|f^{(n)}| - f^{(n)}\},$$

and therefore (the last inequality follows from (7.7))

$$0 \leqslant E_{n-1}(f) - \int_{-1}^{1} V_n f^{(n)} \leqslant \int_{|t| \geqslant \delta_n} V_n |f^{(n)}|$$

$$\leqslant d_{n,0} \int_{\delta_n}^{1} \tilde{V}_n(t)(1-t^2)^{-n/2}\, dt \leqslant d_{n,0} \frac{2^{1-n}}{n!} (1-\delta_n^2)^{n/4}.$$

Remark. If $I = [-1, 1]$ is replaced by $[a, b]$, then Theorem 7(a) reads as follows

$$E_{n-1}(f\colon [a, b])$$
$$= \frac{4}{n!} (L/2)^{n+1} \cdot (|f^{(n)}(c)| + O(1/n)\{L\,|f^{(n+1)}(c)| + L^2 D_{n,1}\}),$$

where

$$c = (a+b)/2; \qquad L = (b-a)/2,$$

and

$$D_{n,1} = \sup_{a \leqslant u \leqslant b} |f^{(n+2)}(u)| \cdot (\sqrt{(b-u)(u-a)}/L)^n.$$

1. EXAMPLE. $f(x) = e^{ax}$. Theorem 7(a) yields

$$2^{n-1} \cdot n!\, E_{n-1}(f) = |a|^n \cdot \{1 + O_a(1/n)\}.$$

Using Theorem 7(b), we obtain

$$2^{n-1} \cdot E_{n-1}(f) = |a|^n \cdot \sum_{l=0}^{\infty} \frac{c_{n,l} \cdot a^{2l}}{(n+2l)!},$$

and Theorem 8 implies, for example,

$$n!\, 2^{n-1} \cdot E_{n-1}(f)$$
$$= |a|^n \cdot \left\{1 + \frac{a^2}{4(n+2)} + \frac{a^4}{32(n+2)(n+3)} + O_a(n^{-3})\right\}.$$

Similarly, we might treat the function $f(x) = \int_0^a e^{xt}\, dg(t)$ with a monotonic function g, and obtain (for example)

$$2^{n-1} \cdot n!\, E_{n-1}(f) = \int_0^a t^n dg(t) \cdot \{1 + O_a(1/n)\}.$$

2. EXAMPLE. $f(x) = \cos \omega x$ with real ω.

Since f is an even function we have $E_{2n-1}(f) = E_{2n-2}(f)$. For $|\omega| \leqslant \pi/2$ we deduce from Theorem 7(b)

$$2^{2n-1} E_{2n-1}(f) = \omega^{2n} \sum_{k=0}^{\infty} c_{2n,k} \frac{(-\omega^2)^k}{(2n+2k)!}, \tag{7.8}$$

and from Theorem 8(a) (with $k = 2$, $\delta_{2n} = 1$, $\gamma_{2n,k} = 0$, $d_{2n,k} = \omega^{2n+4}$; observe (7.5) and $\omega^4 = O(\omega^2)$),

$$\frac{2^{2n-1}(2n)!}{\omega^{2n}} E_{2n-1}(f) = 1 - \omega^2 n^{-1}/8 + O(\omega^2/n^2). \tag{7.9}$$

For $|\omega| > \pi/2$ we use Theorem 8(b) (with $\delta_{2n} = \pi/(2\omega)$, and $d_{2n,0} = \omega^{2n}$) or Theorem 8(a) (with $k = 2$, $\delta_{2n} = \pi/(2\omega)$, $\gamma_{2n,k} = 0$, $d_{2n,k} = \omega^{2n+4}$) and this leads to an extra error term of the form (put $q = 1 - (\pi/(2\omega))^2$) $((\omega^{2n})/(2n)!) \cdot O(q^{n/2})$ in (7.8), and $O(q^{n/2} + \omega^4 n^{-2})$ in (7.9).

We remark that (7.8) does not hold true for all ω, since, for example, for $n = 1$ the value of the right hand side of (7.8) is

$$\omega^2 \cdot \int_{-1}^{1} V_1(t) \cos \omega t\, dt = 2\omega^2 \int_0^1 (1 - t) \cos \omega\, t\, dt = 2(1 - \cos \omega),$$

and especially 0 if $\omega = 2\pi$.

3. EXAMPLE. $f(x) = e^{ax^2}$ with $a \neq 0$.

According to Lemma 4 this function is in the Markoff-class M_{2n}. We apply Theorem 7(b) and obtain (let $m \approx n^{1/2}$)

$$\pm 2^{2n-1} \cdot E_{2n-1}(f) = a^n \cdot \sum_{k=0}^{\infty} c_{2n,k} \frac{a^k}{(n+k)!}$$

$$= a^n \cdot \left\{ \sum_{k=0}^{m-1} \cdots + \sum_{k=m}^{\infty} \cdots \right\} = a^n \{I + II\}.$$

Since $((2n+2k)!)/(4^k(2n+k)!\,(n+k)!)$ decreases in k, (7.3) implies $(c_{2n,k})/((n+k)!) \leqslant 8/(k!n!)$, and therefore $n! \cdot |II| \leqslant 8 \cdot \sum_{k=m}^{\infty} (|a|^k)/(k!) = O_a(1/n)$. From (7.5) we deduce for $k \leqslant n^{1/2}$

$$n!\,2^k k! \cdot \frac{c_{2n,k}}{(n+k)!}$$
$$= \frac{(2n+1)(2n+k+1)(2n+k+2)(2n+k+3)\cdots(2n+2k)}{2^k(n+1)n+2)\cdots(n+k)(2n+k+1)}$$
$$\times \{1+O(2^{-n})\} = 1 + O\left(\frac{k^2}{n}\right),$$

and therefore

$$n! \cdot I = \sum_{k=0}^{m-1} \frac{a^k}{2^k k!}(1+O(k^2/n)),$$
$$= \exp(a/2) + O(1)\left\{(|a|/2)^m \frac{1}{m!} e^{|a|/2} + \frac{1}{n}\sum_{k=0}^{\infty} \frac{k^2|a|^k}{2^k k!}\right\}$$
$$= \exp(a/2) + O_a(1/n).$$

Thus, we obtained

$$E_{2n-2}(f) = E_{2n-1}(f) = \frac{2}{n!}(|a|/4)^n \{\exp(a/2) + O_a(1/n)\}.$$

4. EXAMPLE. $f(x) = x^m$ with $m > n$.

Similarly to (7.1), we have $E_{n-2}(x^{n+2k}) = E_{n-1}(x^{n+2k})$. Now x^{n+2k} is in the Markoff-class M_n, and therefore

$$E_{n-1}(x^{n+2k}) = \frac{(n+2k)!}{(2k)!}\int_{-1}^{1} V_n(t) \cdot t^{2k} = 2^{1-n} \cdot c_{n,k}.$$

Using (7.4), we obtain the order of $E_{n-1}(x^{n+2k})$, and using (7.5) we get $E_{n-1}(x^{n+2k})$ with an exponentially small relative error, if $2k \leqslant n$ (see [7] and [8] for the case of L^∞-approximation). Using Theorem 6 one could obtain $E_{n-1}(x^{n+2k})$ up to a relative error of the form $O(1/n)$ for all k.

5. EXAMPLE. $f(x) = (x-t)_+^{n-1}/(n-1)!$ for fixed $t \in I$.

We have (by Theorem 2 or by Peano's theorem) $E_{n-1}(f) = V_n(t)$. Thus the order of $E_{n-1}(f)$ is given by $\tilde{V}_n(t)$.

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